ANALYTIC SMOOTHNESS EFFECT OF SOLUTIONS FOR SPATIALLY HOMOGENEOUS LANDAU EQUATION

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Abstract. In this paper, we study the smoothness effect of Cauchy problem for the spatially homogeneous Landau equation in the hard potential case and the Maxwellian molecules case. We obtain the analytic smoothing effect for the solutions under rather weak assumptions on the initial datum.

1. Introduction

In this paper we study the Cauchy problem for the following spatially homogeneous Landau equation

$$\begin{cases} \partial_t f = \nabla_v \cdot \{ \int_{\mathbb{R}^3} a(v-v_*)[f(v_*)\nabla_v f(v) - f(v)\nabla_v f(v_*)]dv_* \}, \\ f(0,v) = f_0(v), \end{cases} \tag{1.1}$$

where $f(t,v) \geq 0$ stands for the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, and (a_{ij}) is a nonnegative symmetric matrix given by

$$a_{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)|v|^{\gamma+2}. \tag{1.2}$$

We only consider here the condition $\gamma \in [0,1]$, which is called the hard potential case when $\gamma \in (0,1]$ and the Maxwellian molecules case when $\gamma = 0$. Set $c = \sum_{i,j=1}^{3} \partial_{v_i v_j} a_{ij} = -2(\gamma+3)\,|v|^{\gamma}$ and

$$\bar{a}_{ij}(t,v) = \left(a_{ij} * f\right)(t,v) = \int_{\mathbb{R}^3} a_{ij}(v-v_*)f(t,v_*)dv_*, \quad \bar{c} = c * f.$$

Then the Cauchy problem (1.1) can be rewritten as

$$\begin{cases} \partial_t f = \sum_{i,j=1}^{3} \bar{a}_{ij}\partial_{v_i v_j} f - \bar{c}f, \\ f(0,v) = f_0(v), \end{cases} \tag{1.3}$$

which is a non-linear diffusion equation, for the coefficients $\bar{a}_{i,j}$ satisfy some uniformly elliptic property (see Proposition 4 in [8]) and $\bar{a}_{ij}, \bar{c}$ depend on the solution f.

The Landau equation can be obtained as a limit of the Boltzmann equation when the collisions become grazing, cf. [5] and references therein for more details. In this paper, we are mainly concerned with the analytic regularity of the solutions for the spatially homogeneous Landau equation, which gives partial support to the conjecture on the smoothness of solutions for the Boltzmann equation with singular (or non cutoff) cross sections. This

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conjecture has been proven in certain particular cases, see [1, 2, 7] for the Sobolev smoothness and [12] for the Gevrey smoothness. Recently, a lot of progress has been obtained on the study of the Sobolev regularity for the solutions of Landau equations, cf [4, 8, 9, 15, 16] and references therein, which shows that in some sense the Landau equation can be seen as a non-linear and non-local analog of the hypo-elliptic Fokker-Planck equation. That means, the weak solutions, once constructed under rather weak assumptions on the initial datum, will become smooth or, even more, rapidly decreasing in v at infinity. In the Gevrey class frame, some results have been obtained on the propagation of regularity for the solutions of the Landau equation or Boltzmann equation (see [3, 6, 14]).

Motivated by the smoothness effect of heat equation, one may expect analytic or even more ultra-analytic regularity in $t > 0$ for the solutions of the Cauchy problem (1.3). Recently, Morimoto-Xu [11] proved the ultra-analytic effect for the Cauchy problem (1.3) of the Maxwellian molecules case, which is understandable since in this particular case the coefficients $a_{i,j}(v)$ are ultra-analytic functions (polynomial functions). Here we shall consider the analytic smoothness effect in the hard potential case, which would be more complicated since in this case the coefficients $a_{i,j}$, given in (1.2), are no longer a polynomial functions and only analytic functions away from the origin. So it is reasonable in this paper to consider the analytic smoothness effects of the Cauchy problem (1.3).

Now we give some notations used throughout the paper. For a mult-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, denote

$$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \quad \alpha! = \alpha_1!\alpha_2!\alpha_3!, \quad \partial^\alpha = \partial_{v_1}^{\alpha_1}\partial_{v_2}^{\alpha_2}\partial_{v_3}^{\alpha_3}.$$

We say $\beta = (\beta_1, \beta_2, \beta_3) \le (\alpha_1, \alpha_2, \alpha_3) = \alpha$ if $\beta_i \le \alpha_i$ for each i. For a multi-index α and a nonnegative integer k with $k \le |\alpha|$, if no confusion occurs, we shall use $\alpha - k$ to denote some multi-index $\bar{\alpha}$ satisfying $\bar{\alpha} \le \alpha$ and $|\bar{\alpha}| = |\alpha| - k$. As in [8], we denote by $M(f(t)), E(f(t))$ and $H(f(t))$ respectively the mass, energy and entropy of the function $f(t, v)$, i.e.,

$$M(f(t)) = \int_{\mathbb{R}^3} f(t, v)\, dv, \quad E(f(t)) = \frac{1}{2}\int_{\mathbb{R}^3} f(t, v)\, |v|^2\, dv,$$

$$H(f(t)) = \int_{\mathbb{R}^3} f(t, v) \log f(t, v)\, dv,$$

and denote $M_0 = M(f(0))$, $E_0 = E(f(0))$ and $H_0 = H(f(0))$. It's known that the solutions of the Landau equation satisfy the formal conservation laws:

$$M(f(t)) = M_0, \quad E(f(t)) = E_0, \quad H(f(t)) \le H_0, \qquad \forall\, t \ge 0.$$

Here we adopt the following notations,

$$\|\partial^\alpha f(t, \cdot)\|_{L^p_s} = \left(\int_{\mathbb{R}^3} \left|\partial^\alpha f(t, v)\right|^p \left(1 + |v|^2\right)^{s/2} dv\right)^{\frac{1}{p}}, \quad p \ge 1,$$

$$\|f(t, \cdot)\|^2_{H^m_s} = \sum_{|\alpha| \le m} \|\partial^\alpha f(t)\|^2_{L^2_s}.$$

In the sequel, for simplicity we always write $\|f(t)\|_{L^p_s}$ instead of $\|f(t, \cdot)\|_{L^p_s}$, etc.

Before stating our main theorem, we recall some related results obtained in [8, 16]. In the hard potential case, the existence, uniqueness and Sobolev regularity of the weak solution had been studied by Desvillettes-Villani (cf. Theorem 5, Theorem 6 and Theorem 7 of [8]), and they proved that, under rather weak assumptions on the initial datum (e.g.

$f_0 \in L^1_{2+\delta}$ with $\delta > 0$,), there exists a weak solution f of the Cauchy problem (1.3) such that for all time $t_0 > 0$, all integer $m \geq 0$, and all $s > 0$,

$$\sup_{t \geq t_0} \|f(t,\cdot)\|_{H^m_s} \leq C,$$

where C is a constant depending only on γ, M_0, E_0, H_0, m, s and t_0. Moreover $f(t,v) \in C^\infty(\mathbb{R}^+_t; \mathcal{S}(\mathbb{R}^3))$, where $\mathbb{R}^+_t\ =]0,+\infty[$ and $\mathcal{S}(\mathbb{R}^3)$ denotes the space of smooth functions which are rapidly decreasing in v at infinity. If $f_0 \in L^2_p$ with $p > 5\gamma + 15$, then the Cauchy problem (1.3) admits a unique smooth solution. In the Maxwellian case, Villani [16] proved that the Cauchy problem (1.3) admits a unique classical solution for any initial datum and for all $t > 0$, $f(t,v)$ is bounded and belong to $C^\infty(\mathbb{R}^3_v)$.

Now let us give some equivalent definition of analytic functions. Let u be a real function defined in $\mathbb{R}^N$. We say u is real analytic in $\mathbb{R}^N$ if $u \in C^\infty(\mathbb{R}^N)$ and there exists a constant C such that for all multi-indices $\alpha \in \mathbb{N}^N$,

$$\|\partial^\alpha u\|_{L^2(\mathbb{R}^N)} \leq C^{|\alpha|}\, |\alpha|!,$$

which is equivalent to

$$e^{c_0(-\triangle_v)^{\frac{1}{2}}} u \in L^2(\mathbb{R}^N)$$

for some constant $c_0 > 0$, where $e^{c_0(-\triangle_v)^{\frac{1}{2}}} u$ is the Fourier multiplier defined by

$$e^{c_0(-\triangle_v)^{\frac{1}{2}}} u = \mathscr{F}^{-1}\left(e^{c_0|\xi|}\hat{u}(\xi)\right).$$

Starting from the smooth solution, we state our main result on the analytic regularity as follows.

Theorem 1.1. *Let f_0 be the initial datum with finite mass, energy and entropy and $f(t,v)$ be any solution of the Cauchy problem (1.3) such that for all time $t_0 > 0$ and all integer $m \geq 0$,*

$$\sup_{t \geq t_0} \|f(t,\cdot)\|_{H^m_\gamma} \leq C \tag{1.4}$$

with C a constant depending only on γ, M_0, E_0, H_0, m and t_0. Then for all time $t > 0$, $f(t,v)$, as a real function of v variable, is analytic in $\mathbb{R}^3_v$. Moreover, for all time $t_0 > 0$, there exists a constant $c_0 > 0$, depending only on M_0, E_0, H_0, γ and t_0, such that for all $t \geq t_0$,

$$\|e^{c_0(-\triangle_v)^{\frac{1}{2}}} f(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq C(t+1),$$

where C is a constant depending only on M_0, E_0, H_0, γ and t_0.

Remark 1.2. *As a consequence, the solutions given in* [8] *are real analytic in $\mathbb{R}^3_v$ for any $t > 0$.*

Remark 1.3. *The result of Theorem 1.1 can be extended to any space dimensional case.*

The plan of the paper is as follows: In section 2 we present the proof of the main result. Section 3 is devoted to the proof of the lemma 2.2 in the section 2 which is crucial to the proof of the main result here.

2. Proof of the main result

This section is devoted to the proof of the main results. To simplify the notations, in the sequel we always use $\sum_{1\leq|\beta|\leq|\mu|}$ to denote the summation over all the multi-indices β satisfying $\beta \leq \mu$ and $1 \leq |\beta| \leq |\mu|$. Likewise $\sum_{1\leq|\beta|\leq|\mu|-1}$ denotes the summation over all the multi-indices β satisfying $\beta \leq \mu$ and $1 \leq |\beta| \leq |\mu| - 1$, etc. We begin with the following lemma.

Lemma 2.1. *For all multi-indices $\mu \in \mathbb{N}^3, |\mu| \geq 2$, we have*

$$\sum_{1\leq|\beta|\leq|\mu|-1} \frac{|\mu|}{|\beta|^4\,(|\mu|-|\beta|)} \leq 24, \tag{2.1}$$

and

$$\sum_{1\leq|\beta|\leq|\mu|-1} \frac{|\mu|}{|\beta|^3\,(|\mu|-|\beta|)^2} \leq 24. \tag{2.2}$$

Proof. For each positive integer l, we denote by $N\{|\beta| = l\}$ the number of the multi-indices β with $|\beta| = l$. In the case when the space dimension equals to 3, one has

$$N\{|\beta| = l\} = \frac{(l+2)!}{2!\,l!} = \frac{1}{2}(l+1)(l+2).$$

Thus we can compute directly

$$\sum_{1\leq|\beta|\leq|\mu|-1} \frac{|\mu|}{|\beta|^4\,(|\mu|-|\beta|)} \leq \sum_{l=1}^{|\mu|-1}\sum_{|\beta|=l} \frac{|\mu|}{l^4(|\mu|-l)} \leq \sum_{l=1}^{|\mu|-1} \frac{3\,|\mu|}{l^2(|\mu|-l)}.$$

Without loss of generality, we may assume $|\mu| - 1$ is an even integer. Then the inequality above can be rewritten as

$$\sum_{1\leq|\beta|\leq|\mu|-1} \frac{|\mu|}{|\beta|^4\,(|\mu|-|\beta|)} \leq \sum_{l=1}^{(|\mu|-1)/2} \frac{3\,|\mu|}{l^2(|\mu|-l)} + \sum_{l=(|\mu|+1)/2}^{|\mu|-1} \frac{3\,|\mu|}{l^2(|\mu|-l)}.$$

In the case when $l \leq \frac{|\mu|-1}{2}$, we have $|\mu| - l \geq \frac{|\mu|+1}{2}$. Then it follows that

$$\sum_{l=1}^{(|\mu|-1)/2} \frac{3\,|\mu|}{l^2(|\mu|-l)} \leq \frac{6\,|\mu|}{|\mu|+1}\sum_{l=1}^{(|\mu|-1)/2} \frac{1}{l^2} < 12.$$

In the case when $l \geq \frac{|\mu|+1}{2}$, then we have

$$\sum_{l=(|\mu|+1)/2}^{|\mu|-1} \frac{3\,|\mu|}{l^2(|\mu|-l)} \leq \frac{12}{|\mu|+1}\sum_{l=(|\mu|+1)/2}^{|\mu|-1} \frac{1}{|\mu|-l} < 12.$$

Combination of the above three inequalities gives the desired inequality (2.1). Similarly we can deduce the inequality (2.2). Lemma 2.1 is proved. □

Next, we give a following crucial lemma, which is important in the proof of the main result. Throughout the paper we always assume $(-i)! = 1$ for nonnegative integer i.

Lemma 2.2. *There exist constants B, C_1 and $C_2 > 0$ with B depending only on the dimension and C_1, C_2 depending only on M_0, E_0, H_0 and γ, such that for all multi-indices*

$\mu \in \mathbb{N}^3$ with $|\mu| \geq 2$ and all $t > 0$, we have

$$\begin{aligned}\frac{d}{dt}\|\partial^\mu f(t)\|^2_{L^2} + C_1\|\nabla_v\partial^\mu f(t)\|^2_{L^2_\gamma} &\leq C_2\,|\mu|^2\,\|\nabla_v\partial^{\mu-1} f(t)\|^2_{L^2_\gamma} + \\ +C_2 \sum_{2\leq|\beta|\leq|\mu|} C^\beta_\mu\|\nabla_v\partial^{\mu-\beta+1} f(t)\|_{L^2_\gamma}&\cdot\|\nabla_v\partial^{\mu-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\beta-2} + \\ +C_2 \sum_{0\leq|\beta|\leq|\mu|} C^\beta_\mu\|\partial^\beta f(t)\|_{L^2_\gamma}&\cdot\|\nabla_v\partial^{\mu-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\mu-\beta},\end{aligned}$$

where $C^\beta_\mu = \frac{\mu!}{(\mu-\beta)!\beta!}$ is the binomial coefficients, $[G(f(t))]_\omega = \|\partial^\omega f(t)\|_{L^2} + B^{|\omega|}\,(|\omega|-3)!$, and $\mu - l$ denotes some multi-index $\tilde\mu$ satisfying $\tilde\mu \leq \mu$ and $|\tilde\mu| = |\mu| - l$.

The proof of Lemma 2.2 will be given in the section 3.

Proposition 2.3. *Let f_0 be the initial datum with finite mass, energy and entropy and $f(t, v)$ be any solution of the Cauchy problem (1.3) satisfying the condition (1.4). Then for any T_0, T_1 with $0 < T_0 < T_1 < +\infty$, there exists a constant A, depending only on M_0, E_0, H_0,γ and T_0, such that for any ρ with $0 < \rho < \min\left\{\frac{1}{4}, (T_1 - T_0)/2\right\}$, and any nonnegative integer m, the following estimate*

$$\sup_{s\in\Omega_\rho} \|\partial^\alpha f(s)\|_{L^2} + \left\{\int_{\Omega_\rho} \|\nabla_v\partial^{\tilde\alpha} f(t)\|^2_{L^2_\gamma}\,dt\right\}^{\frac{1}{2}} \leq \frac{C_{T_0,T_1}\,A^{m+1}}{\rho^m}[(m-2)!] \tag{2.3}$$

holds for any multi-indices α, $\tilde\alpha$ with $|\alpha| = |\tilde\alpha| = m$, where the interval Ω_ρ and the constant C_{T_0,T_1} are given by

$$\Omega_\rho = [T_0 + \rho, T_1 - \rho], \qquad C_{T_0,T_1} = 2(T_1 - T_0 + 1).$$

Remark 2.4. *Note that the constant A in (2.3) is independent of T_1, which can be deduced from the fact that all H^m_γ norms of f are bounded uniformly in time (see the assumption (1.4) above). In fact, the constant A can be calculated explicitly. This can be seen in the process of following proof.*

Proof of Proposition 2.3. For any ρ with $0 < \rho < \min\left\{\frac{1}{4}, (T_1 - T_0)/2\right\}$, we define $\Omega_{\rho,j}$, $j \geq 1$, by setting

$$\Omega_{\rho,j} = [t_j + \rho, t_j + 1 - \rho]$$

with $t_j = T_0 + \frac{j-1}{2}$. Observe $\rho < \frac{1}{4}$, then we can find a positive integer N_0 with $N_0 \leq 2(T_1 - T_0 + 1)$, such that

$$\Omega_\rho = [T_0 + \rho, T_1 - \rho] \subset \bigcup_{j=1}^{N_0} \Omega_{\rho,j}.$$

Hence the desired estimate (2.3) will follow if we can find a constant A, depending only on T_0, M_0, E_0, H_0 and γ, such that for any ρ with $0 < \rho < \min\left\{\frac{1}{4}, (T_1 - T_0)/2\right\}$, and any nonnegative integer m, the following estimate

$$\sup_{s\in\Omega_{\rho,j}} \|\partial^\alpha f(s)\|_{L^2} + \left\{\int_{\Omega_{\rho,j}} \|\nabla_v\partial^{\tilde\alpha} f(t)\|^2_{L^2_\gamma}\,dt\right\}^{\frac{1}{2}} \leq \frac{A^{m+1}}{\rho^m}[(m-2)!] \tag{2.4}$$

holds for all multi-indices α, $\tilde\alpha$ with $|\alpha| = |\tilde\alpha| = m$, and all $1 \leq j \leq N_0$.

We shall use induction on m to prove the estimate (2.4). First, we take a constant A large enough such that

$$A \geq 2\sup_{s\geq T_0} \|f(s)\|_{H^2_\gamma} + 2. \tag{2.5}$$

In view of (1.4), we see that the constant A depends only on T_0, M_0, E_0, H_0 and γ. Observing that $|\Omega_{\rho,j}|$, the Lebesgue measure of $\Omega_{\rho,j}$, is less than 1 and that

$$\sup_{s\in\Omega_{\rho,j}} \|f(s)\|_{H^2_\gamma} \le \sup_{s\ge t_j} \|f(s)\|_{H^2_\gamma} \le \sup_{s\ge T_0} \|f(s)\|_{H^2_\gamma} \le \frac{A}{2}, \quad 1 \le j \le N_0,$$

we compute, for any $\gamma, \tilde\gamma$ with $|\gamma| = |\tilde\gamma| \le 1$,

$$\sup_{s\in\Omega_{\rho,j}} \|\partial^\gamma f(s)\|_{L^2} + \left\{\int_{\Omega_{\rho,j}} \|\nabla_v \partial^{\tilde\gamma} f(t)\|^2_{L^2_\gamma}\, dt\right\}^{\frac12} \le \frac{A}{2} + \frac{A}{2} \le A, \quad 1 \le j \le N_0,$$

which implies that the estimate (2.4) holds for $m = 0, 1$.

We assume, for some integer $k \ge 2$, the estimate (2.4) holds for all m with $m \le k-1$. We now need to prove the validity of (2.4) for $m = k$, or equivalently, to show the following two estimates: for any $0 < \rho < \min\left\{\frac14,\ (T_1 - T_2)/2\right\}$,

$$\sup_{s\in\Omega_{\rho,j}} \|\partial^\alpha f(s)\|_{L^2} \le \frac12 \frac{A^{|\alpha|+1}}{\rho^{|\alpha|}}[(|\alpha|-2)!], \quad \forall\ |\alpha| = k, \tag{2.6}$$

and

$$\left\{\int_{\Omega_{\rho,j}} \|\nabla_v \partial^{\tilde\alpha} f(t)\|^2_{L^2_\gamma} dt\right\}^{\frac12} \le \frac12 \frac{A^{|\tilde\alpha|+1}}{\rho^{|\tilde\alpha|}}[(|\tilde\alpha|-2)!], \quad \forall\ |\tilde\alpha| = k. \tag{2.7}$$

In the following discussion, we fix j, ρ, α and $\tilde\alpha$, with $1 \le j \le N_0$, $0 < \rho < \min\left\{\frac14, (T_1 - T_0)/2\right\}$ and $|\alpha| = |\tilde\alpha| = k$. In this case we introduce a cut-off function $\varphi(t)$ which is a smooth function with compact support in $\Omega_{\tilde\rho,j}$, where $\tilde\rho = \frac{k}{k+1}\rho$, and satisfies $0 \le \varphi \le 1$ and $\varphi = 1$ in $\Omega_{\rho,j}$. It is easy to see that

$$\sup_{t\in\mathbb{R}} \left|\frac{d\varphi(t)}{dt}\right| \le \bar{C}\, k/\rho, \tag{2.8}$$

where the constant $\bar{C}$ is independent of k and ρ, and

$$\frac{1}{\rho^k} \le \frac{1}{\tilde\rho^k} = \frac{1}{\rho^k} \times (\frac{k+1}{k})^k \le \frac{3}{\rho^k}. \tag{2.9}$$

First we prove the estimate (2.6). By using Lemma 2.2, one has

$$\begin{aligned}
&\frac{d}{dt}\|\partial^\alpha f(t)\|^2_{L^2} + C_1 \|\nabla_v \partial^\alpha f(t)\|^2_{L^2_\gamma} \le C_2\, |\alpha|^2\, \|\nabla_v \partial^{\alpha-1} f(t)\|^2_{L^2_\gamma} + \\
&+ C_2 \sum_{2\le|\beta|\le|\alpha|} C_\alpha^\beta \|\nabla_v \partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma} \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\beta-2} + \\
&+ C_2 \sum_{0\le|\beta|\le|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\alpha-\beta}.
\end{aligned}$$

Rewriting the last term of the right hand side as

$$\begin{aligned}
&C_2 \|f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_\alpha \\
&+ C_2 \sum_{1\le|\beta|\le|\alpha|} C_\alpha^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\alpha-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\alpha-\beta},
\end{aligned}$$

we obtain that

$$\begin{aligned}
&\frac{d}{dt}\|\partial^\alpha f(t)\|^2_{L^2} + C_1\|\nabla_v\partial^\alpha f(t)\|^2_{L^2_\gamma} \le C_2\,|\alpha|^2\,\|\nabla_v\partial^{\alpha-1} f(t)\|^2_{L^2_\gamma} + \\
&+C_2\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_\alpha + \\
&+C_2\sum_{2\le|\beta|\le|\alpha|} C^\beta_\alpha\|\nabla_v\partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma}\|\nabla_v\partial^{\alpha-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\beta-2} + \\
&+C_2\sum_{1\le|\beta|\le|\alpha|} C^\beta_\alpha\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\alpha-\beta}.
\end{aligned}$$

Multiplying by the cut-off function $\varphi(t)$ in the both sides of the inequality above to get

$$\begin{aligned}
&\frac{d}{dt}\left(\varphi(t)\|\partial^\alpha f(t)\|^2_{L^2}\right) + C_1\varphi(t)\|\nabla_v\partial^\alpha f(t)\|^2_{L^2_\gamma} \\
&\le \frac{d\varphi}{dt}\cdot\|\partial^\alpha f(t)\|^2_{L^2} + C_2\cdot\varphi(t)\,|\alpha|^2\,\|\nabla_v\partial^{\alpha-1} f(t)\|^2_{L^2_\gamma} + \\
&+C_2\cdot\varphi(t)\,\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_\alpha + \\
&+C_2\varphi(t)\sum_{2\le|\beta|\le|\alpha|} C^\beta_\alpha\|\nabla_v\partial^{\alpha-\beta+1} f(t)\|_{L^2_\gamma}\|\nabla_v\partial^{\alpha-1} f(t)\|_{L^2_\gamma}[G(f(t))]_{\beta-2} + \\
&+C_2\cdot\varphi(t)\sum_{1\le|\beta|\le|\alpha|} C^\beta_\alpha\|\partial^\beta f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\alpha-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\alpha-\beta}.
\end{aligned}$$

In the sequel, we set

$$[G(f)]_{\rho,\beta} = \sup_{t\in\Omega_{\rho,j}}[G(f(t))]_\beta = \sup_{t\in\Omega_{\rho,j}}\|\partial^\beta f(t)\|_{L^2} + B^{|\beta|}(|\beta|-3)!. \tag{2.10}$$

Since supp $\varphi\subset\Omega_{\tilde\rho,j}$ with $\tilde\rho=\frac{k\rho}{k+1}$, and $\varphi(t)=1$ for all $t\in\Omega_{\rho,j}$ and $\varphi(t_j)=0$, then for any $s\in\Omega_{\rho,j}$, we integrate the inequality above over the interval $[t_j,s]\subset[t_j,t_j+1-\rho]$ to get, from Cauchy inequality, that

$$\begin{aligned}
\|\partial^\alpha f(s)\|^2_{L^2} &= \varphi(s)\|\partial^\alpha f(s)\|^2_{L^2} - \varphi(t_j)\,\|\partial^\alpha f(t_j)\|^2_{L^2} \\
&\le (S_1)+(S_2)+(S_3)+(S_4)+(S_5),
\end{aligned}$$

where (S_j), $1\le j\le 5$, are given by

$$(S_1) = \sup_{t\in\mathbb{R}}\left|\frac{d\varphi}{dt}\right|\int_{\Omega_{\tilde\rho,j}}\|\partial^\alpha f(t)\|^2_{L^2}dt;$$

$$(S_2) = C_2\,|\alpha|^2\int_{\Omega_{\tilde\rho,j}}\|\nabla_v\partial^{\alpha-1} f(t)\|^2_{L^2_\gamma}dt;$$

$$(S_3) = C_2\sup_{t\in\Omega_{\tilde\rho,j}}\|f(t)\|_{L^2_\gamma}\cdot\left\{\int_{\Omega_{\tilde\rho,j}}[G(f(t))]^2_\alpha dt\right\}^{\frac12}\left\{\int_{\Omega_{\tilde\rho,j}}\|\nabla_v\partial^{\alpha-1} f(t)\|^2_{L^2_\gamma}dt\right\}^{\frac12};$$

$$\begin{aligned}
(S_4) &= C_2\sum_{2\le|\beta|\le|\alpha|} C^\beta_\alpha[G(f)]_{\tilde\rho,\,\beta-2}\left\{\int_{\Omega_{\tilde\rho,j}}\|\nabla_v\partial^{\alpha-\beta+1} f(t)\|^2_{L^2_\gamma}dt\right\}^{\frac12}\times \\
&\quad\times\left\{\int_{\Omega_{\tilde\rho,j}}\|\nabla_v\partial^{\alpha-1} f(t)\|^2_{L^2_\gamma}dt\right\}^{\frac12};
\end{aligned}$$

and

$$
\begin{aligned}
(S_5) \quad = \quad & C_2 \sum_{1\le|\beta|\le|\alpha|} C_\alpha^\beta [G(f)]_{\tilde\rho,\alpha-\beta} \left\{\int_{\Omega_{\tilde\rho,j}} \|\partial^\beta f(t)\|^2_{L^2_\gamma} dt\right\}^{\frac12} \times \\
& \times \left\{\int_{\Omega_{\tilde\rho,j}} \|\nabla_v \partial^{\alpha-1} f(t)\|^2_{L^2_\gamma} dt\right\}^{\frac12}.
\end{aligned}
$$

In order to treat these terms, we need the following estimates which can be deduced directly from the the induction hypothesis. The validity of (2.4) for all $m \le k-1$ implies that

$$
\left\{\int_{\Omega_{\tilde\rho,j}} \|\nabla_v \partial^\gamma f(t)\|^2_{L^2_\gamma} dt\right\}^{\frac12} \le \frac{A^{|\gamma|+1}}{\tilde\rho^{|\gamma|}}[(|\gamma|-2)!], \quad 0 \le |\gamma| \le k-1; \tag{2.11}
$$

$$
\sup_{t\in\Omega_{\tilde\rho,j}} \|\partial^\lambda f(t)\|_{L^2} \le \frac{A^{|\lambda|+1}}{\tilde\rho^{|\lambda|}}((|\lambda|-2)!), \quad 0 \le |\lambda| \le k-1; \tag{2.12}
$$

and

$$
\left\{\int_{\Omega_{\tilde\rho,j}} \|\partial^\beta f(t)\|^2_{L^2_\gamma} dt\right\}^{1/2} \le \frac{A^{|\beta|}}{\tilde\rho^{|\beta|-1}}[(|\beta|-3)!], \quad 1 \le |\beta| \le |\alpha|, \tag{2.13}
$$

the last inequality following from the fact $\|\partial^\beta f\|_{L^2_\gamma} \le \|\nabla_v \partial^{\beta-1} f\|_{L^2_\gamma}$ for any multi-index β with $1 \le |\beta| \le |\alpha|$. Consequently, if we take A large enough such that $A \ge B$, then in view of (2.10), (2.12) and (2.13), one has

$$
[G(f)]_{\tilde\rho,\lambda} \le \frac{2A^{|\lambda|+1}}{\tilde\rho^{|\lambda|}}((|\lambda|-2)!), \quad 0 \le |\lambda| \le |\alpha|-1, \tag{2.14}
$$

and

$$
\left\{\int_{\Omega_{\tilde\rho,j}} [G(f(t))]^2_\alpha dt\right\}^{1/2} \le \frac{2A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-3)!]. \tag{2.15}
$$

Now we are ready to handle the terms (S_j) for $1 \le j \le 5$. In the process below, the notations C_ℓ, $\ell \ge 3$, will be used to denote different constants which are larger than 1 and depend only on M_0, E_0, H_0, γ and T_0. Observe $\|\partial^\alpha f(s)\|_{L^2} \le \|\partial^\alpha f(s)\|_{L^2_\gamma}$, one has, by (2.8) and (2.13),

$$
(S_1) \le \frac{C_3 k}{\rho}\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-3)!]\right\}^2 \le C_4\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|}}[(|\alpha|-2)!]\right\}^2, \tag{2.16}
$$

where we used the fact that $\rho^{-1} < \tilde\rho^{-1} < \tilde\rho^{-2}$ and that $|\alpha| = k$. Next, by virtue of (2.11), we obtain

$$
(S_2) \le C_2 |\alpha|^2 \left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-3)!]\right\}^2 \le C_5\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]\right\}^2. \tag{2.17}
$$

To estimate the term (S_3), we use the estimates (2.5), (2.11) and (2.15), which gives

$$
(S_3) \le C_6 A\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-3)!]\right\}^2. \tag{2.18}
$$

Now we handle the term (S_4). By virtue of (2.11) and (2.14), we can deduce that the term (S_4) is bounded by

$$\sum_{2\leq|\beta|\leq|\alpha|}\frac{C_2\,|\alpha|!}{|\beta|!(|\alpha|-|\beta|)!}\frac{A^{|\beta|-1}}{\tilde\rho^{|\beta|-2}}[(|\beta|-4)!]\frac{A^{|\alpha|-|\beta|+2}}{\tilde\rho^{|\alpha|-|\beta|+1}}[(|\alpha|-|\beta|-1)!]\times$$
$$\times\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-3)!],$$

that is, from the estimate (2.1) of Lemma 2.1, we have

$$\begin{aligned}(S_4) &\leq C_7A\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]\right\}^2\times\left\{\sum_{2\leq|\beta|\leq|\alpha|-1}\frac{|\alpha|}{|\beta|^4\,(|\alpha|-|\beta|)}+1\right\}\\ &\leq C_8A\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]\right\}^2. \qquad (2.19)\end{aligned}$$

Similarly, by virtue of (2.2), (2.11), (2.13) and (2.14), we can get the estimate for the term (S_5), i.e.

$$(S_5)\leq C_9A\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|-1}}[(|\alpha|-2)!]\right\}^2. \qquad (2.20)$$

Combining the estimates (2.16)-(2.20), one has, for any $s\in\Omega_{\rho,j}$, that

$$\begin{aligned}\|\partial^\alpha f(s)\|_{L^2}^2 &\leq \sum_{i=1}^5(S_i)\leq C_{10}A\left\{\frac{A^{|\alpha|}}{\tilde\rho^{|\alpha|}}[(|\alpha|-2)!]\right\}^2\\ &\leq C_{11}A\left\{\frac{A^{|\alpha|}}{\rho^{|\alpha|}}[(|\alpha|-2)!]\right\}^2, \qquad (2.21)\end{aligned}$$

the last inequality above follows from the estimate (2.9). Taking A large enough such that

$$A\geq 4\max\left\{\sup_{s\geq T_0}\|f(s)\|_{H^2_\gamma}+1,\ B,\ C_{11}\right\},$$

then we obtain finally

$$\|\partial^\alpha f(s)\|_{L^2}^2\leq\left\{\frac{1}{2}\frac{A^{|\alpha|+1}}{\rho^{|\alpha|}}[(|\alpha|-2)!]\right\}^2,\quad \forall\ s\in\Omega_{\rho,j},$$

which gives the proof of the estimate (2.6).

Now, it remains to prove the estimate (2.7), which can be handled similarly as the proof of the estimate (2.6). Let us apply Lemma 2.2 again with $\mu=\tilde\alpha$, and then we multiply the

cut-off function $\varphi(t)$ in the both sides of the estimate in Lemma 2.2 to get

$$\begin{aligned}
&\frac{d}{dt}\left(\varphi(t)\|\partial^{\tilde\alpha} f(t)\|_{L^2}^2\right) + C_1\varphi(t)\|\nabla_v\partial^{\tilde\alpha} f(t)\|_{L^2_\gamma}^2 \\
&\leq \frac{d\varphi}{dt}\cdot\|\partial^{\tilde\alpha} f(t)\|_{L^2}^2 + C_2\cdot\varphi(t)\,|\tilde\alpha|^2\,\|\nabla_v\partial^{\tilde\alpha-1} f(t)\|_{L^2_\gamma}^2 + \\
&+C_2\cdot\varphi(t)\,\|f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\tilde\alpha-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\tilde\alpha} + \\
&+C_2\varphi(t)\sum_{2\leq|\beta|\leq|\tilde\alpha|} C_{\tilde\alpha}^{\beta}\|\nabla_v\partial^{\tilde\alpha-\beta+1} f(t)\|_{L^2_\gamma}\|\nabla_v\partial^{\tilde\alpha-1} f(t)\|_{L^2_\gamma}[G(f(t))]_{\beta-2} + \\
&+C_2\cdot\varphi(t)\sum_{1\leq|\beta|\leq|\tilde\alpha|} C_{\tilde\alpha}^{\beta}\|\partial^{\beta} f(t)\|_{L^2_\gamma}\cdot\|\nabla_v\partial^{\tilde\alpha-1} f(t)\|_{L^2_\gamma}\cdot[G(f(t))]_{\tilde\alpha-\beta} \\
&\stackrel{\text{def}}{=}\mathcal{N}(t).
\end{aligned}$$

Observe supp $\varphi\subset\Omega_{\tilde\rho,j}$, then integrating the above inequality over the interval $\Omega_{\tilde\rho,j}$ yields that

$$C_1\int_{\Omega_{\tilde\rho,j}}\varphi(s)\|\nabla_v\partial^{\tilde\alpha} f(s)\|_{L^2_\gamma}^2 ds \leq \int_{\Omega_{\tilde\rho,j}}\mathcal{N}(t)dt.$$

Repeating the previous arguments we used to estimate the terms (S_1)-(S_5), one has

$$\int_{\Omega_{\tilde\rho,j}}\mathcal{N}(t)dt \leq C_{11}A\left\{\frac{A^{|\alpha|}}{\rho^{|\alpha|}}[(|\alpha|-2)!]\right\}^2,$$

where C_{11} is the same constant as appeared in (2.21). From these inequalities, together with the fact that $A\geq 4C_{11}$ and

$$\int_{\Omega_{\rho,j}}\|\nabla_v\partial^{\tilde\alpha} f(s)\|_{L^2_\gamma}^2 ds \leq \int_{\Omega_{\tilde\rho,j}}\varphi(s)\|\nabla_v\partial^{\tilde\alpha} f(s)\|_{L^2_\gamma}^2 ds,$$

we can deduce that the estimate (2.7) holds. The proof of Proposition 2.3 is completed.

□

Now we present the proof of the main result.

Proof of Theorem 1.1. Given $t_0>0$, and for any $t\geq t_0$, we take $T_0=\frac{3t_0}{4}$, $T_1=t+\frac{3t_0}{4}$ and $\rho=\frac{t_0}{4}$ in the estimate (2.3). This gives, for any $|\alpha|=m\geq 0$,

$$\|\partial^\alpha f(t)\|_{L^2}\leq\sup_{s\in[T_0+\rho,T_1-\rho]}\|\partial^\alpha f(s)\|_{L^2}\leq\frac{2(t+1)(4A)^{m+1}}{t_0^m}[(m-2)!],$$

with the constant A depending only on M_0, E_0, H_0, γ and t_0. Thus one has

$$\frac{c_0^m}{m!}\sum_{|\alpha|=m}\|\partial^\alpha f(t)\|_{L^2}\leq 16A(t+1)\left(\frac{1}{2}\right)^m,$$

where $c_0=\frac{t_0}{16A}$. Hence we have

$$\|e^{c_0(-\triangle_v)^{\frac12}} f(t,\cdot)\|_{L^2(\mathbb{R}^3)}\leq\tilde A(t+1),$$

where the constant $\tilde A$ depends only on M_0, E_0, H_0, γ and t_0. Theorem 1.1 is proved. □

3. Proof of Lemma 2.2

This section is devoted to the proof of Lemma 2.2. Our starting point is the following uniformly ellipticity property of the matrix $(\overline{a}_{ij})$, cf. Proposition 4 in [8].

Lemma 3.1. *There exists a constant K, depending only on γ and M_0, E_0, H_0, such that*

$$\sum_{1\leq i,j\leq 3} \overline{a}_{ij}(t,v)\xi_i\xi_j \geq K(1+|v|^2)^{\gamma/2}\,|\xi|^2\,,\quad \forall\,\xi\in\mathbb{R}^3. \tag{3.1}$$

Remark 3.2. *Although the ellipticity of $(\overline{a}_{ij})$ was proved in* [8] *in the hard potential case $\gamma>0$, it's still true for the Maxwellian case $\gamma=0$. This can be seen in the proof of Proposition 4 in* [8].

Lemma 3.3. *There exists a constant L, depending only on the space dimension, such that for any fixed positive integer N, $N\geq 2$, one can find a function $\psi_N\in C_0^\infty(\mathbb{R}^3)$ with compact support in $\left\{v\in\mathbb{R}^3 \mid |v|\leq 2\right\}$, satisfying that $0\leq\psi_N(v)\leq 1$ and $\psi_N(v)=1$ on the ball $\left\{v\in\mathbb{R}^3 \mid |v|\leq 1\right\}$, and*

$$\sup\left|\partial^\lambda\psi_N\right|\leq (LN)^{|\lambda|},\quad \forall\,\lambda,\ |\lambda|\leq N. \tag{3.2}$$

Proof. For the construction of ψ_N, we refer to [10, 13]. Choose a non-negative function $\rho\in C_0^\infty(\mathbb{R}^3)$ with compact support in $\{v\in\mathbb{R}^3 \mid |v|\leq\frac{1}{2}\}$, and $\int_{\mathbb{R}^3}\rho(v)dv=1$. Set

$$L=\max_{|\alpha|\leq 1}\int_{\mathbb{R}^3}\left|\partial^\alpha\rho(v)\right|dv.$$

Let χ be the characteristic function of the set $\left\{v\in\mathbb{R}^3 \mid |v|\leq\frac{3}{2}\right\}$. For each $r\geq 0$, defined

$$\rho_r(v)=r^{-3}\rho(v/r).$$

Set then

$$\psi_N=\chi*\rho_{1/N}*\cdots*\rho_{1/N}$$

with N factors $\rho_{1/N}$. Direct verification shows that the function ψ_N satisfies the desired properties. □

Lemma 3.4. *There exists a constant B, depending only on the dimension, such that for all multi-indices β with $|\beta|\geq 2$ and all $g,h\in L^2_\gamma(\mathbb{R}^3)$, one has*

$$\textstyle\sum_{1\leq i,j\leq 3}\int_{\mathbb{R}^3}(\partial^\beta\overline{a}_{ij}(t,v))g(v)h(v)dv\leq C\|g\|_{L^2_\gamma}\|h\|_{L^2_\gamma}[G(f(t))]_{\beta-2},\quad \forall\,t>0,$$

where $[G(f(t))]_{\beta-2}=\left\{\|\partial^{\beta-2}f(t)\|_{L^2}+B^{|\beta|-2}(|\beta|-5)!\right\}$.

Proof. Let L be the constant given in Lemman 3.3, and let $\psi=\psi_{|\beta|}\in C_0^\infty(\mathbb{R}^3)$ be the function constructed in Lemma 3.3 for $N=|\beta|$. Write $a_{ij}=\psi a_{ij}+(1-\psi)a_{ij}$. Then $\overline{a}_{ij}=(\psi a_{ij})*f+[(1-\psi)a_{ij}]*f$, and hence

$$\partial^\beta\overline{a}_{ij}=[\partial^{\tilde\beta}(\psi a_{ij})]*(\partial^{\beta-\tilde\beta}f)+\left\{\partial^\beta[(1-\psi)a_{ij}]\right\}*f,$$

where $\tilde\beta$ is an arbitrary multi-index satisfying $\tilde\beta\leq\beta$ and $|\tilde\beta|=2$.

We first treat the term $[\partial^{\tilde\beta}(\psi a_{ij})]*(\partial^{\beta-\tilde\beta}f)$. It is easy to verify that for all $\tilde\beta$ with $\left|\tilde\beta\right|=2$,

$$\left|(\partial^{\tilde\beta}a_{ij})(v-v_*)\right|\leq C\,|v-v_*|^\gamma\,,$$

thus we can compute

$$\begin{aligned}\left|[\partial^{\tilde\beta}(\psi a_{ij})] * (\partial^{\beta-\tilde\beta} f)(v)\right| &= \left|\int_{\mathbb{R}^3} [\partial^{\tilde\beta}(\psi a_{ij})](v-v_*)\cdot(\partial^{\beta-\tilde\beta} f)(v_*)dv_*\right|\\ &\leq C\int_{\{|v_*-v|\leq 2\}}\left|(\partial^{\beta-\tilde\beta} f)(v_*)\right| dv_*\\ &\leq C\|\partial^{\beta-2} f(t)\|_{L^2}.\end{aligned}$$

Here the notation C is used to denote different constants which will depend only on the γ, M_0, E_0 and H_0.

In the next step we treat the term $\left\{\partial^\beta[(1-\psi)a_{ij}]\right\} * f$. By using Leibniz's formula, one has

$$\begin{aligned}&\left|(\partial^\beta[(1-\psi)a_{ij}]) * f(v)\right|\\ &= \Big|\sum_{0\leq|\lambda|\leq|\beta|} C_\beta^\lambda \int_{\mathbb{R}^3}[\partial^{\beta-\lambda}(1-\psi)](v-v_*)\cdot\left(\partial^\lambda a_{ij}\right)(v-v_*)\cdot f(t,v_*)dv_*\Big|\\ &\leq J_1+J_2,\end{aligned}$$

where

$$J_1=\sum_{0\leq|\lambda|\leq|\beta|-1} C_\beta^\lambda\int_{\{1\leq|v_*-v|\leq 2\}}\left|\partial^{\beta-\lambda}\psi(v-v_*)\right|\cdot\left|\partial^\lambda a_{ij}(v-v_*)\right| f(t,v_*)dv_*,$$

and

$$J_2=\int_{\{|v_*-v|\geq 1\}}|1-\psi(v-v_*)|\cdot\left|\partial^\beta a_{ij}(v-v_*)\right|\cdot f(t,v_*)dv_*.$$

In view of (1.2), we can find a constant $\tilde C$, such that for all multi-indices $\lambda\leq\beta$,

$$\left|\left(\partial^\lambda a_{ij}\right)(v-v_*)\right|\leq \tilde C^{|\lambda|}\,|\lambda|!,\quad \text{for } 1\leq|v_*-v|\leq 2,$$

which along with the estimate (3.2) gives

$$\begin{aligned}J_1 &\leq L^{|\beta|}(|\beta|)^{|\beta|}\cdot\|f(t)\|_{L^1}\sum_{0\leq|\lambda|\leq|\beta|-1}\left(\frac{\tilde C}{L}\right)^{|\lambda|}\\ &\leq 30^{|\beta|}L^{|\beta|}(|\beta|-5)!\|f(t)\|_{L^1}\sum_{0\leq|\lambda|\leq|\beta|-1}\left(\frac{\tilde C}{L}\right)^{|\lambda|},\end{aligned}$$

the last estimate follows from the fact that

$$|\beta|^{|\beta|}\leq e^{|\beta|}\,|\beta|!\leq 30^{|\beta|}(|\beta|-5)!.$$

Furthermore, it is easy to see that, for all β with $|\beta|\geq 2$,

$$\left|\left(\partial^\beta a_{ij}\right)(v-v_*)\right|\leq \tilde C^{|\beta|}\,|\beta|!(1+|v_*|^\gamma+|v|^\gamma),\quad \text{for } |v_*-v|\geq 1.$$

Hence

$$J_2\leq 2\tilde C^{|\beta|}\,|\beta|!\cdot\|f(t)\|_{L^1_\gamma}(1+|v|^\gamma)\leq 30^{|\beta|}\tilde C^{|\beta|}(|\beta|-5)!\cdot\|f(t)\|_{L^1_\gamma}(1+|v|^\gamma).$$

So we may let $L\geq 2\tilde C$ and then take B large enough such that $B\geq 30L$, thus it follows from the estimates above that

$$J_1+J_2\leq(50L)^2\|f(t)\|_{L^1_\gamma}B^{|\beta|-2}(|\beta|-5)!(1+|v|^2)^{\gamma/2}.$$

This along with the fact $\|f(t)\|_{L^1_\gamma} \le M_0 + 2E_0$ gives that

$$\left|(\partial_v^\beta[(1-\psi)a_{ij}]) * f(v)\right| \le J_1 + J_2 \le CB^{|\beta|-2}(|\beta|-5)!(1+|v|^2)^{\gamma/2}.$$

Combining the estimate on the term $[\partial^{\tilde\beta}(\psi a_{ij})] * (\partial^{\beta-\tilde\beta} f)$ yields that

$$\begin{aligned}\left|\partial^\beta \overline{a}_{ij}(v)\right| &\le C\left\{\|\partial^{\beta-2} f(t)\|_{L^2} + B^{|\beta|-2}[(|\beta|-5)!]\cdot(1+|v|^2)^{\gamma/2}\right\}\\ &\le C[G(f(t))]_{\beta-2}\cdot(1+|v|^2)^{\gamma/2}.\end{aligned}$$

By using the Cauchy's inequality and the estimates above, one can deduce the desired inequality in Lemma 3.4. □

Similarly, we have following estimate

Lemma 3.5. *For all multi-indices β with $|\beta| \ge 0$ and all $g, h \in L^2_\gamma(\mathbb{R}^3)$, one has*

$$\int_{\mathbb{R}^3}(\partial^\beta \overline{c}(t,v))g(v)h(v)dv \le C\|g\|_{L^2_\gamma}\|h\|_{L^2_\gamma}\cdot[G(f(t))]_\beta, \quad \forall\, t \ge 0.$$

Next, we give the proof of Lemma 2.2.

Proof of Lemma 2.2. Set $b_j = \sum_{1\le i\le 3}\partial_{v_i}a_{ij}(v) = -2\,|v|^\gamma\, v_j$, thus we have $\sum_{1\le i\le 3}\partial_{v_i}\overline{a}_{ij}(v) = \overline{b}_j(v)$, $\sum_{1\le j\le 3}\partial_{v_j}\overline{b}_j = \overline{c}$. Since the solution f satisfies

$$\partial_t f = \sum_{1\le i,j\le 3}\overline{a}_{ij}\partial_{v_iv_j} f - \overline{c} f,$$

thus we have

$$\begin{aligned}\frac{d}{dt}\|\partial^\mu f(t)\|^2_{L^2} &= 2\int_{\mathbb{R}^3}[\partial_t\partial^\mu f(t,v)]\cdot[\partial^\mu f(t,v)]dv\\
&= 2\sum_{1\le i,j\le 3}\int_{\mathbb{R}^3}[\partial^\mu(\overline{a}_{ij}\partial_{v_iv_j} f - \overline{c}f)]\cdot[\partial^\mu f(t,v)]dv\\
&= 2\sum_{1\le i,j\le 3}\int_{\mathbb{R}^3}\overline{a}_{ij}\left(\partial_{v_iv_j}\partial^\mu f\right)\cdot(\partial^\mu f)\,dv +\\
&\quad +2\sum_{1\le i,j\le 3}\sum_{|\beta|=1}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^\beta\overline{a}_{ij}\right)\left(\partial_{v_iv_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\,dv +\\
&\quad +2\sum_{1\le i,j\le 3}\sum_{|\beta|=2}^{|\mu|}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^\beta\overline{a}_{ij}\right)\left(\partial_{v_iv_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\,dv -\\
&\quad -2\sum_{0\le|\beta|\le|\mu|}C^\beta_\mu\int_{\mathbb{R}^3}\left(\partial^{\mu-\beta}\overline{c}\right)\left(\partial^\beta f\right)\cdot(\partial^\mu f)\,dv\\
&= (I)+(II)+(III)+(IV).\end{aligned}$$

We shall estimate the each term above by following steps.

Step 1. Upper bound for the term (I).

Integrating by parts, one has

$$\begin{aligned}(I) &= -2\sum_{1\le i,j\le 3}\int_{\mathbb{R}^3}\overline{a}_{ij}\left(\partial_{v_j}\partial^\mu f\right)\cdot(\partial_{v_i}\partial^\mu f)\,dv - 2\sum_{1\le j\le 3}\int_{\mathbb{R}^3}\overline{b}_j\left(\partial_{v_j}\partial^\mu f\right)\cdot(\partial^\mu f)\,dv\\
&= (I)_1 + (I)_2.\end{aligned}$$

The ellipticity property (3.1) of $(\overline{a}_{ij})_{i,j}$ gives that

$$(I)_1 \le -2K \int_{\mathbb{R}^3} \left|\nabla_v \partial^\mu f\right|^2 (1+|v|^2)^{\gamma/2} dv = -2K \|\nabla_v \partial^\mu f(t)\|^2_{L^2_\gamma}.$$

For the term $(I)_2$, we integrate by parts to get

$$(I)_2 = -(I)_2 + 2\int_{\mathbb{R}^3} \overline{c}\,(\partial^\mu f)\cdot(\partial^\mu f)\,dv.$$

This along with the fact

$$|\overline{c}(v)| \le C\|f(t)\|_{L^1_\gamma}(1+|v|^2)^{\gamma/2} \le C(1+|v|^2)^{\gamma/2}$$

shows immediately

$$(I)_2 \le C\|\partial^\mu f(t)\|^2_{L^2_\gamma} \le C\|\nabla_v \partial^{\mu-1} f(t)\|^2_{L^2_\gamma}.$$

Combining these estimates, we get the upper bound for the term (I), i.e.

$$(I) \le -2K\|\nabla_v \partial^\mu f(t)\|^2_{L^2_\gamma} + C\|\nabla_v \partial^{\mu-1} f(t)\|^2_{L^2_\gamma}. \tag{3.3}$$

Step 2. Upper bound for the term (II).

Recall $(II) = 2\sum_{1\le i,j\le 3} \sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^\beta \overline{a}_{ij}\right)\left(\partial_{v_i v_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\,dv$. Integrating by parts, we get

$$\begin{aligned}(II) &= -2\sum_{1\le j\le 3}\sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^\beta \overline{b}_j\right)\left(\partial_{v_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\,dv - \\ &\quad -2\sum_{1\le i,j\le 3}\sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^\beta \overline{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta} f\right)\cdot(\partial_{v_i}\partial^\mu f)\,dv \\ &= (II)_1 + (II)_2.\end{aligned}$$

Note $\left|\partial^\beta \overline{b}_j(t,v)\right| \le C(1+|v|^2)^{\gamma/2}$ for any β with $|\beta|=1$ and hence

$$(II)_1 \le C\,|\mu|\cdot\|\nabla_v \partial^{\mu-\beta} f(t)\|_{L^2_\gamma}\|\partial^\mu f(t)\|_{L^2_\gamma} \le C\,|\mu|\cdot\|\nabla_v\partial^{\mu-1} f(t)\|^2_{L^2_\gamma}.$$

For the term $(II)_2$, since $\mu = \beta + (\mu-\beta)$, it can be rewritten as following form

$$(II)_2 \quad = -2\sum_{1\le i,j\le 3}\sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^\beta \overline{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta} f\right)\cdot\left(\partial^\beta \partial_{v_i}\partial^{\mu-\beta} f\right)dv.$$

Since $|\beta| = 1$, we can integrate by parts to get

$$\begin{aligned}(II)_2 &= 2\sum_{1\le i,j\le 3}\sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^\beta \overline{a}_{ij}\right)\left(\partial_{v_j}\partial^\mu f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta} f\right)dv + \\ &\quad +2\sum_{1\le i,j\le 3}\sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^{\beta+\beta} \overline{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta} f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta} f\right)dv \\ &= -(II)_2 + 2\sum_{1\le i,j\le 3}\sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^{\beta+\beta} \overline{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta} f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta} f\right)dv.\end{aligned}$$

Hence

$$(II)_2 \quad = \sum_{1\le i,j\le 3}\sum_{|\beta|=1} C_\mu^\beta \int_{\mathbb{R}^3} \left(\partial^{\beta+\beta} \overline{a}_{ij}\right)\left(\partial_{v_j}\partial^{\mu-\beta} f\right)\cdot\left(\partial_{v_i}\partial^{\mu-\beta} f\right)dv.$$

This along with the fact $\left|\partial^{\beta+\beta}\overline{a}_{ij}(v)\right| \leq C(1+|v|^2)^{\gamma/2}$ for all β with $|\beta|=1$ gives that

$$(II)_2 \leq C \sum_{|\beta|=1} C_\mu^\beta \cdot \|\nabla_v \partial^{\mu-\beta} f\|^2_{L^2_\gamma} \leq C\,|\mu| \cdot \|\nabla_v \partial^{\mu-1} f\|^2_{L^2_\gamma}.$$

Thus we obtain

$$(II) \leq C\,|\mu| \cdot \|\nabla_v \partial^{|\mu|-1} f\|^2_{L^2_\gamma}. \tag{3.4}$$

Step 3. Upper bound for the terms (III) and (IV) and the conclusion.

Recall $(III) = 2\sum_{1\leq i,j\leq 3}\sum_{|\beta|=2}^{|\mu|} C_\mu^\beta \int_{\mathbb{R}^3}\left(\partial^\beta \overline{a}_{ij}\right)\left(\partial_{v_i v_j}\partial^{\mu-\beta} f\right)\cdot(\partial^\mu f)\,dv$, and

$$(IV) = -2\sum_{0\leq|\beta|\leq|\mu|} C_\mu^\beta \int_{\mathbb{R}^3}\left(\partial^{\mu-\beta}\overline{c}\right)\left(\partial^\beta f\right)\cdot(\partial^\mu f)\,dv.$$

By virtue of Lemma 3.4 and lemma 3.5, it follows that

$$\begin{aligned}(III) &\leq C\sum_{i,j=1}^{3}\sum_{|\beta|=2}^{|\mu|} C_\mu^\beta \|\partial_{v_i v_j}\partial^{\mu-\beta} f(t)\|_{L^2_\gamma} \cdot \|\partial^\mu f(t)\|_{L^2_\gamma}[G(f(t))]_{\beta-2} \\ &\leq C\sum_{|\beta|=2}^{|\mu|} C_\mu^\beta \|\nabla_v \partial^{\mu-\beta+1} f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\mu-1} f(t)\|_{L^2_\gamma}[G(f(t))]_{\beta-2},\end{aligned} \tag{3.5}$$

and

$$(IV) \leq C\sum_{0\leq|\beta|\leq|\mu|} C_\mu^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\mu-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\mu-\beta}. \tag{3.6}$$

Combining the estimates (3.3)-(3.6), we can deduce the desired estimate in Lemma 2.2, that is

$$\begin{aligned}&\frac{d}{dt}\|\partial^\mu f(t)\|^2_{L^2} + C_1\|\nabla_v \partial^\mu f(t)\|^2_{L^2_\gamma} \leq C_2\,|\mu|^2\,\|\nabla_v \partial^{\mu-1} f(t)\|^2_{L^2_\gamma} + \\ &+C_2\sum_{2\leq|\beta|\leq|\mu|} C_\mu^\beta \|\nabla_v \partial^{\mu-\beta+1} f(t)\|_{L^2_\gamma}\|\nabla_v \partial^{\mu-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\beta-2} + \\ &+C_2\sum_{0\leq|\beta|\leq|\mu|} C_\mu^\beta \|\partial^\beta f(t)\|_{L^2_\gamma} \cdot \|\nabla_v \partial^{\mu-1} f(t)\|_{L^2_\gamma} \cdot [G(f(t))]_{\mu-\beta},\end{aligned}$$

where C_1, C_2 are two constants depending only on M_0, E_0, H_0 and γ. This completes the proof of Lemma 2.2.

□

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